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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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ROTATING BODIES AND THEIR LIGHTCURVES

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ABSTRACT

This report discusses the theoretical impossibility of deducing the surface reflectivity of a rotating illuminated body from its lightcurve. We first derive the lightcurve for an arbitrary convex body as an expansion in spherical harmonics. From the form of the resulting lightcurve and its Fourier series the negative result follows immediately.

CONTENTS

I.   INTRODUCTION

The analysis of the lightcurve of a rotating body is a topic of long standing in astronomy. The natural satellites of the other planets and the minor planets have been investigated in this fashion. The primary aim has been to deduce the surface reflectivity from the lightcurve so that a deeper analysis of the object may be successful. There is a result due to Russell[*] concerning the impossibility of deducing the surface reflectivity from the lightcurve. For the natural satellites this constraint is of little importance because the most sophisticated model would be an airless triaxial ellipsoid. Data acquired by spacecraft have shown that for some moons this is not true, but then we have the flyby data to aid in our analysis.

In the case of artificial satellites, particularly those covered by solar cells, high order symmetries may exist in the surface reflectivity. As we shall see below these may be impossible to discover from only the lightcurve. We expand on Russell's result and fill in some gaps in his analysis. We also derive the Fourier series (in time) of the lightcurve.

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[*]H. N. Russell, Astrophys. J. 24, 1, (1906).

II. A SPHERICAL BODY

Let r, θ, ϕ be a spherical coordinate system with origin at the center of the sphere: r is the distance from the origin, θ is the colatitude, and ϕ is the azimuth. Let $B(\theta,\phi)$ be the intrinsic brightness[*] of an element of area $dS(\theta,\phi)$ of the surface. Let $\gamma(\theta,\phi)$ be the angle between the outward normal to $d\underline{S}$ and the line of sight to the observer. Finally, let $R(\theta,\phi)$ be the distance between dS and the observer. The amount of light received by the observer from dS is its brightness multiplied by its apparent area divided by the square of its distance; viz.,

$$dL = B\cos\gamma\, dS/R^2. \tag{1a}$$

The total light received by the observer is the sum of all such contributions for which $\cos\gamma > 0$ (e.g., that dS is visible),

$$L = \int B\cos\gamma\, H(\cos\gamma)\, dS/R^2, \tag{1b}$$

where $H(u)$ is the Heaviside function

$$H(u) = \begin{cases} 1 & u > 0 \\ 0 & u < 0 \end{cases}. \tag{2}$$

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[*] The magnitude is related to the brightness by $m = -2.5\log(B)$ + constant.

If the radius of the sphere is a and a<<min[R(θ,φ)] then
θ,φ

$$L \simeq (1/R^2) \int B \cos\gamma H(\cos\gamma) dS, \tag{3}$$

where R is now the distance from the center of the sphere to the observer and $\cos\gamma = \underline{r} \cdot \underline{R} /(rR)$. The vector $\underline{R} = (R,\Theta,\Phi)$ where Θ is the observer's colatitude and Φ is his azimuth (the vertex of γ is now at the origin). Since

$$\cos\gamma = \cos\theta\cos\Theta + \sin\theta\sin\Theta\cos(\phi-\Phi), \tag{4}$$

and

$$dS(\theta,\phi) = a^2 \sin\theta d\theta d\phi, \tag{5}$$

$$L = (a^2/R^2) \int_0^\pi d\theta \int_0^{2\pi} B(\theta,\phi) \cos\gamma H(\cos\gamma) \sin\theta d\phi. \tag{6}$$

Clearly $L = a^2$ (function of Θ,Φ)$/R^2$.

To pursue the analysis further we now assume that $B(\theta,\phi)$ is non-negative, everywhere bounded, has a finite number of extrema, and a finite number of points or curves of discontinuity. These conditions (known as Dirichlet conditions) pose no practical limitations on the form of B. They do assure us that the series expansion for B, in a series of spherical harmonics, converges. For reference we define the spherical harmonics $\{Y_{\ell m}(\theta,\phi)\}$ in terms of the associated Legendre functions $\{P_\ell^m(\cos\theta)\}$ and exponentials $\{\exp(im\phi)\}$ by

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$$Y_{\ell m}(\theta,\phi) = \left[\frac{2\ell+1}{4\pi}\ \frac{\Gamma\ (\ell-m+1)}{\Gamma\ (\ell+m+1)}\right]^{1/2} P_\ell^m\ (\cos\theta)\exp(im\phi), \tag{7a}$$

$$P_\ell^m\ (u) = \frac{(-1)^m(1-u^2)^{m/2}}{2^\ell\Gamma(\ell+1)}\ \cdot\ \frac{d^{\ell+m}(u^2-1)^\ell}{du^{\ell+m}}\ , \tag{7b}$$

$$= (-1)^m(1-u^2)^{m/2} d^m P_\ell(u)/du^m, \tag{7c}$$

$$P_\ell^{-m}(u) = (-1)^m\Gamma(\ell-m+1)\ P_\ell^m(u)/\Gamma(\ell+m+1), \tag{7d}$$

$$P_\ell^m\ (-u) = (-1)^{\ell+m} P_\ell^m(u), \tag{7e}$$

$$P_\ell^m(1) = \delta_{mo} = \begin{cases}1 \text{ if } m=0 \\ 0 \text{ otherwise}\end{cases}, \tag{7f}$$

$$Y_{\ell-m}(\theta,\phi) = (-1)^m Y_{\ell m}^*(\theta,\phi), \tag{7g}$$

where $\Gamma\ (u)$ is the gamma function and $P_\ell\ (u)$ is the Legendre polynomial of order ℓ.

We write $B\ (\theta,\phi)$ as

$$B(\theta,\phi) = \sum_{\ell=0}^\infty\ \sum_{m=-\ell}^\ell B_{\ell m} Y_{\ell m}(\theta,\phi), \tag{8a}$$

where

$$B_{\ell m} = \int_o^\pi d\theta \int_o^{2\pi} \sin\theta Y_{\ell m}^*(\theta,\phi) B(\theta,\phi) d\phi, \tag{8b}$$

and the asterisk denotes complex conjugation. We substitute

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expression (8a) into Eq. (6) and deal with the Heaviside function by expressing $Y_{\ell m}(\theta,\phi)$ in terms of γ and λ. The auxiliary angle λ is the position angle of dS, relative to the origin, as seen by the observer after the surface is projected onto the plane of the sky. From the properties of the rotation group of the spherical harmonics we can write

$$Y_{\ell m}(\theta,\phi) = \sum_{n=-\ell}^{\ell} A_{\ell n} Y_{\ell n}(\gamma,\lambda). \tag{9a}$$

However, from Eq. (4), if $\theta = \Theta$ and $\phi = \Phi$, $\gamma = 0$. From Eq. (7f) it now follows that

$$Y_{\ell m}(\Theta,\Phi) = A_{\ell o}. \tag{9b}$$

We now use Eqs. (6, 8a, 9a, and 9b) to write the ℓ'th order term for L as [cf. Eq. (11)]

$$L_\ell = (a^2/R^2) \sum_{m=-\ell}^{\ell} B_{\ell m} \int_0^{2\pi} d\lambda \int_0^{\pi/2} \sin\gamma\cos\gamma \, [Y_{\ell m}(\Theta,\Phi) Y_{\ell o}(\gamma,\lambda)$$

$$+ \sum_{n=-\ell}^{\ell} A_{\ell n} Y_{\ell n}(\gamma,\lambda)(1-\delta_{no})]d\gamma. \tag{10a}$$

From Eq. (7a) we see that the result of the λ intergral is $2\pi\delta_{no}$ so

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$$L_\ell = (2\pi a^2/R^2) \sum_{m=-\ell}^{\ell} B_{\ell m} Y_{\ell m}(\Theta,\Phi) \int_0^{\pi/2} \sin\gamma\cos\gamma Y_{\ell o}(\gamma,\lambda) d\gamma, \qquad (10b)$$

where

$$Y_{\ell o} = \left[\frac{2\ell+1}{4\pi}\right]^{1/2} P_\ell^o(\cos\gamma) = \left[\frac{2\ell+1}{4\pi}\right]^{1/2} P_\ell(\cos\gamma). \qquad (10c)$$

If we set I_ℓ equal to the integral in Eq. (10b) then

$$L_\ell = (2\pi a^2 I_\ell/R^2) \sum_{m=-\ell}^{\ell} B_{\ell m} Y_{\ell m}(\Theta,\Phi), \qquad (10d)$$

and,

$$L = \sum_{\ell=o}^{\infty} L_\ell = (2\pi a^2/R^2) \sum_{\ell=o}^{\infty} I_\ell \sum_{m=-\ell}^{\ell} B_{\ell m} Y_{\ell m}(\Theta,\Phi). \qquad (11)$$

Equation (11) is our final result giving L as a function of the $\{B_{\ell m}\}$ and the observer's location $\underline{R}$.

The last step is the evaluation of the integrals $\{I_\ell\}$. This is done by using the Rodrigues formula for the Legendre poly-nomials [cf. Eqs. (7b, 7c)] and the change of variable $u = \cos\gamma$. We find

$$\left[\frac{4\pi}{2\ell+1}\right]^{1/2} I_\ell = \int_0^1 u P_\ell(u)\,du = \begin{cases} 1/2 & \text{if } \ell = 0 \\ 1/3 & \text{if } \ell = 1 \\ 1/8 & \text{if } \ell = 2 \\ 0 & \text{if } \ell \text{ is odd and } \geq 3 \\ \dfrac{(-1)^{\frac{\ell}{2}+1}(\ell-3)!!}{(\ell+2)!!} & \text{if } \ell \text{ is even and } \geq 4, \end{cases} \tag{12}$$

where the double factorial is defined by

$$\left. \begin{aligned} (2k)!! &= 2^k \Gamma(k+1) \\ (2k+1)!! &= \Gamma(2k+2)/[2^k \Gamma(k+1)] \end{aligned} \right\} k = 0,\ 1,\ 2,\ldots \tag{13}$$

There are two conclusions we can immediately draw from Eq. (12); first for non-zero values of I_ℓ

$$|I_\ell| \leq \frac{1}{\ell(\ell+2)} \leq \frac{1}{\ell^2}\ \forall\ \ell \geq 2, \tag{14}$$

so, since the series for B [Eq. (8a)] converges, the series for L [Eq. (11)] converges uniformly and absolutely. Hence, L is a continuous function of $\underline{R}$ and $\nabla_{\underline{R}} L$ is continuous even if B is discontinuous. Secondly, since all of the higher order odd harmonics are absent, there exists a denumerable set of different reflectivities all of which yield the identical light curve.

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III. EXTENSION TO ANY CONVEX SURFACE

The simplest two-dimensional surface to discuss is one whose equation is given in the form $z = f(x,y)$. However, we will use the parametric representation $\underline{r} = [x(\theta,\phi), y(\theta,\phi), z(\theta,\phi)]$. We assume (i) $\underline{r}(\theta,\phi)$ is a single valued function, (ii) $\partial\underline{r}/\partial\theta$ and $\partial\underline{r}/\partial\phi$ both exist and are continuous, and (iii) $\frac{\partial\underline{r}}{\partial\theta} \times \frac{\partial\underline{r}}{\partial\phi}$ is not null (i.e., the vector cross product). The element of arc-length, ds, on the surface is given by

$$ds^2 = d\underline{r} \cdot d\underline{r}, \quad d\underline{r} = \frac{\partial\underline{r}}{\partial\theta}\,d\theta + \frac{\partial\underline{r}}{\partial\phi}\,d\phi, \tag{15a}$$

or,

$$ds^2 = Ed\theta^2 + 2Fd\theta d\phi + Gd\phi^2, \tag{15b}$$

where

$$E = \frac{\partial\underline{r}}{\partial\theta} \cdot \frac{\partial\underline{r}}{\partial\theta}, \tag{16a}$$

$$F = \frac{\partial\underline{r}}{\partial\theta} \cdot \frac{\partial\underline{r}}{\partial\phi} \quad (= 0 \text{ here}), \tag{16b}$$

$$G = \frac{\partial\underline{r}}{\partial\phi} \cdot \frac{\partial\underline{r}}{\partial\phi}. \tag{16c}$$

The element of area, dS, of the surface is proportional to

$|\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}|$. Since, for any two vectors $\underline{u}$ and $\underline{v}$,

$$|\underline{u} \times \underline{v}|^2 = |\underline{u}|^2|\underline{v}|^2 - (\underline{u} \cdot \underline{v})^2, \tag{17}$$

dS can be written as

$$dS = (EG-F^2)^{1/2}d\theta d\phi. \tag{18}$$

Finally, define a quantity C (which is essentially a curvature) by

$$C = r^2 \sin\theta/(EG - F^2)^{1/2}. \tag{19}$$

Then the expression for L is

$$L = \int_0^{2\pi}\int_0^{\pi}(B/C)\cos\gamma H(\cos\gamma)\sin\theta d\phi d\theta/R^2, \tag{20}$$

in analogy with Eq. (1b). If $\max_{\theta,\phi}(|\underline{r}|^2) << \min_{\theta,\phi}(|\underline{R}|^2)$ then we may again bring the factor of a^2/R^2 outside of the integral, redefining R and γ as in §II, and proceed exactly as before except the $\{B_{\ell m}\}$ are now interpreted as the expansion coefficients of B/C. Only if C vanishes will our previous results be invalid. The restriction to convex surfaces precludes this possibility.

We might inquire as to the lightcurve of an arbitrary convex body with a constant surface brightness. From the discussion above the answer is again given by Eq. (11) (after reinterpretating the $\{B_{\ell m}\}$) except there can be no $\ell = 1$ term. This follows from the parity of the spherical harmonics, Eq. (7g).

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IV. THE FOURIER SERIES FOR THE LIGHTCURVE

As the orientation of the r, θ, ϕ coordinate system is at our disposal we now choose the polar axis to be the axis of uniform rotation. If the body is precessing then the polar axis is the instantaneous axis of rotation. The observer's colatitude is a constant equal to the inclination of the axis of rotation to the line of sight. The observer's azimuth is simply $\phi = \omega(t - t_o)$ where ω is the angular speed of rotation and t_o is the instant of a prime meridian crossing. Using this in Eq. (11) and rearranging terms (which is permissible because the series converges absolutely) we obtain

$$L = (2\pi a^2/R^2)\{C_o(\theta)/2 + \sum_{n=1}^{\infty} C_n(\theta)\cos[n\omega(t - t_o)]$$

$$+ \sum_{n=1}^{\infty} S_n(\theta)\sin[n\omega(t - t_o)]\}, \tag{21}$$

with

$$C_o(\theta) = 2\sum_{k=o}^{\infty} I_k B_{ko} P_k(\cos\theta), \tag{22a}$$

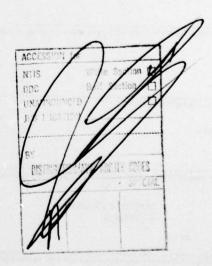$$C_n = \sum_{k=n}^{\infty} I_k[B_{kn}P_k^n(\cos\theta) + B_{k-n}P_k^{-n}(\cos\theta)], \quad n>o \tag{22b}$$

$$S_n = \sum_{k=n}^{\infty} iI_k[B_{kn}P_k^n(\cos\theta) - B_{k-n}P_k^{-n}(\cos\theta)], \quad n>o. \tag{22c}$$
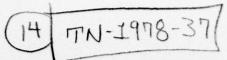
From Eqs. (7g, 8b), $B_{k-n} = (-1)^n B_{kn}^*$.

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V. THE INVERSE PROBLEM

As we mentioned above there exist an infinite number of different B's all of which yield the same L. Clearly then, given L it is impossible to determine B uniquely. Any additional physical constraints we might want to impose on B (i.e., its non-negativity, continuity, various symmetry properties, etc.) can serve to limit our ignorance but not overcome it. The problem is simply indeterminate. As this is the case when a complete set of functions is used the situation will be even worse if some other, incomplete basis is used.

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